Regularity and geometric character of solution of a degenerate parabolic equation

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Abstract: This work studies the regularity and the geometric significance of solution of the Cauchy problem for a degenerate parabolic equation $u_t = \Delta u^m$. Our main objective is to improve the Hölder estimate obtained by pioneers and then, to show the geometric characteristic of free boundary of degenerate parabolic equation. To be exact, the present work will show that:

(1) the weak solution $u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}^n \times \mathbb{R}^+)$, where $\alpha \in (0,1)$ when $m \geq 2$ and $\alpha = 1$ when $m \in (1,2)$;

(2) the surface $\phi = (u(x,t))^\beta$ is a complete Riemannian manifold, which is tangent to $\mathbb{R}^n$ at the boundary of the positivity set of $u(x,t)$.

(3) the function $(u(x,t))^\beta$ is a classical solution to another degenerate parabolic equation if β is large sufficiently;

Moreover, some explicit expressions about the speed of propagation of $u(x,t)$ and the continuous dependence on the nonlinearity of the equation are obtained.

Recalling the older Hölder estimate ($u(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\mathbb{R}^n \times \mathbb{R}^+)$ with $0 < \alpha < 1$ for all $m > 1$), we see our result (1) improves the older result and, based on this conclusion, we can obtain (2), which shows the geometric characteristic of free boundary.

Keywords : degenerate parabolic equation; regularity; geometric character; Hölder estimate

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1 Introduction

Consider the Cauchy problem of nonlinear parabolic equation

$$\begin{cases} u_t = \Delta u^m & \text{in } Q, \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $Q = \mathbb{R}^n \times \mathbb{R}^+, n \geq 1, m > 1$ and

$$0 \leq u_0(x) \leq M, \qquad 0 < \int_{\mathbb{R}^n} u_0(x)dx < \infty. \tag{1.2}$$

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The equation in (1.1) is an example of nonlinear evolution equations and many interesting results, such as the existence, uniqueness, continuous dependence on the nonlinearity of the equation and large time behavior are obtained during the past several decades. By a weak solution of (1.1), (1.2) in Q, we mean a nonnegative function $u(x,t)$ such that, for any given $T > 0$,

$$\int_0^T \int_{\mathbb{R}^n} \left(u^2 + |\nabla u^m|^2\right) dxdt < \infty$$

and

$$\int_0^T \int_{\mathbb{R}^n} (\nabla u^m \cdot \nabla f - u f_t)\, dxdt = \int_{\mathbb{R}^n} u_0(x) f(x,0) dx$$

for any continuously differentiable function $f(x,t)$ with compact support in $\mathbb{R}^n \times (0,T)$.

We know that (see [3, 4, 5, 6, 8, 12, 13, 14]) the Cauchy problem (1.1),(1.2) permits a unique weak solution $u(x,t)$ which has the following properties:

$$0 \le u(x,t) \le M, \tag{1.3}$$

$$\int_{\mathbb{R}^n} u(x,t) dx = \int_{\mathbb{R}^n} u_0(x) dx, \tag{1.4}$$

$$\frac{\partial u}{\partial t} \ge \frac{-u}{(m-1)t}, \tag{1.5}$$

$$\Delta\left(\frac{m}{m-1} u^{m-1}\right) \ge \frac{-n}{n(m-1)+2} \cdot \frac{1}{t} \tag{1.6}$$

$$\|u - v\|_{L^2(Q_T)} \le C \max_{s \in [0,M]} \left| s^{\frac{1}{j}} - s^{\frac{m}{j}} \right|, \tag{1.7}$$

where $j = 1, 2, 3...$ and v is the solution to the Cauchy problem of linear heat equation with the same initial value

$$\begin{cases} v_t = \Delta v & \text{in } Q, \\ v(x,0) = u_0(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.8}$$

Moreover, the solution $u(x,t)$ can be obtained (see [9, 15]) as a limit of solutions $u_\eta (\eta \longrightarrow 0^+)$ of the Cauchy problem

$$\begin{cases} u_t = \Delta u^m & \text{in } Q, \\ u(x,0) = u_0(x) + \eta & \text{on } \mathbb{R}^n, \end{cases} \tag{1.9}$$

and the solutions u_η is taken in the classical sense. We know that D.G.Aronson, Ph.Benilan (see theorem 2, p.104 in [8]) claimed that: if u is the weak solution to the Cauchy problem (1.1) with the initial value (1.2), then $u \in C(Q)$ and $u \ge 0$; J. L. Vazquez (see Proposition 6 in Ch.2 of [13]) proved $u \in C^\infty(Q_+)$, where

$$Q_+ = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ : \ u(x,t) > 0\}.$$

Before this, the same conclusion was established by A. Friedman (see theorem 11 and corollary 2 in Ch.3 [10]). Moreover, employing so called *bootstrap argument*, D.G.Aronson, B.H.Gilding

and L. A. Peletier (see [2, 4, 5, 6]) also claimed $u \in C^\infty(Q_+)$ with more details. Therefore, we can divide the space-time $\mathbb{R}^n \times \mathbb{R}^+$ into two parts: $Q = Q_+ \cup Q_0$, where

$$Q_0 = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ :\ u(x,t) = 0\}.$$

Furthermore, if Q_0 contains an open set, say, Q_1, we can also obtain $u(x,t) \in C^\infty(Q_1)$ owing to $u(x,t) \equiv 0$ in Q_1. Thereby, we may suspect that the solution of degenerate parabolic equation is actually smooth in Q except a set of measure 0. In order to improve the regularity of $u(x,t)$, many authors have made hard effort in this direction. The earliest contribution to the subject was made, maybe, by D.G. Aronson and B.H.Gilding and L.A.Peleiter(see [2, 4]). They proved that the solution to the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} & \text{in } \mathbb{R}^1 \times \mathbb{R}^+,\ m > 1, \\ u(x,0) = u_0(x) & \text{on } \mathbb{R}^1 \end{cases} \tag{1.10}$$

is continuous in $\mathbb{R}^1 \times (0,+\infty)$ if the nonnegative initial value satisfies a good condition. Moreover, if the initial value $0 \le u_0 \le M$ and u_0^m is Lip-continuous, $u(x,t)$ can be continuous on $\mathbb{R}^1 \times [0,+\infty)$ (see [2]). As to the case of $n \ge 1$, L.Caffarelli and A.Friedman (see [15]), proved that the solution $u(x,t)$ to the Cauchy problem (1.1), (1.2) is Hölder continuous in Q :

$$|u(x,t) - u(x_0,t_0)| \le C\left(|x-x_0|^\alpha + |t-t_0|^{\frac{\alpha}{2}}\right) \tag{$*$}$$

for some $0 < \alpha < 1$ and $C > 0$. Moreover, for the general equation $u_t = \nabla \cdot (u\nabla p)$, $p = \kappa(u)$, L .Caffarelli and J.L.Vazques ([16]) established the property of finite propagation and the persistence of positivity, where κ may be a general operator. To study this problem more precisely, D. G. Aronson, S.B. Angenent and J. Graveleau (see [7, 20]) constructed a interesting radially symmetric solution $u(r,t)$ to the focusing problem for the equation of (1.1). Denoting the porous medium pressure $V = \frac{m}{m-1}u^{m-1}$, they claimed $V = Cr^\delta$ at the fusing time, where $0 < \delta < 1, C$ is a positive constant. Moreover,

$$\lim_{r \downarrow 0} \frac{V(r,\eta r^\alpha)}{r^{2-\alpha}} = \frac{\varphi(c^*\eta)}{-\eta}.$$

To study the regularity of the weak solution of (1.1), (1.2), the present work will show the following more precise conclusion: for every $h \in (m-1,m)$, there exists a $C > 0$ such that

$$|u(x,t) - u(x_0,t_0)| \le C\left(|x-x_0|^{\frac{1}{h}} + |t-t_0|^{\frac{1}{2h}}\right), \tag{1.11}$$

where,

$$h = \begin{cases} 1 & \text{if } \ 1 < m < 2, \\ h \in (m-1,m) & \text{if } \ m \ge 2. \end{cases}$$

We see that the range of $\frac{1}{h}$ is $(0,1]$ not $(0,1)$, thereby, the older Hölder estimate $(*)$ is improved by (1.11).

Moreover, we will show that the functions $\frac{\partial u^\beta}{\partial x_i}$ are continuous if β is large sufficiently because we can employ (1.11) to obtain

$$|u^\beta(x,t) - u^\beta(x_*,t_*)| \le C\left(|x - x_*|^{\frac{\beta}{h}} + |t - t_*|^{\frac{\beta}{2h}}\right)$$

for another positive constant C. By this inequality, we want to get the continuous partial derivatives $\frac{\partial u^\beta}{\partial t}$ and $\frac{\partial u^\beta}{\partial x_i}$ $i = 1, 2, ..., n$, that is to say,

$$\phi(x,t) \in C^1(\mathbb{R}^n) \tag{1.12}$$

for every given $t > 0$, where $\phi(x,t) = u^\beta(x,t)$. In particular, we will prove that the function $\phi(x,t)$ satisfies the degenerate parabolic equation

$$\frac{\partial \phi}{\partial t} = m\left[\phi^{\frac{m-1}{\beta}}\Delta\phi + \frac{m-\beta}{\beta}\phi^{\frac{m-\beta-1}{\beta}}|\nabla\phi|^2\right]$$

in the classical sense.

For every fixed $t > 0$, we define a n-dimensional surface $S(t)$, which floats in the space $\mathbb{R}^{n+1}$ with the time t:

$$S(t):\begin{cases} x_i = x_i, & i = 1, 2, 3, ..., n, \\ x_{n+1} = \phi(x,t), & \end{cases}$$

where the function $\phi(x,t)$ is mentioned above. We will discuss the geometric character of $S(t)$. We know that Y.Giga and R.V.Kohn studied the fourth-order total variation flow and the fourth-order surface diffusion law (see [22]), and proved that the solution becomes identically zero in finite time. To be exact, the solution surface will coincide with $\mathbb{R}^n$ in finite time. Because this phenomenon will never occur for our surface $S(t)$ owing to (1.2) and (1.4), so we will discuss the relationship between $S(t)$ and $\mathbb{R}^n$.

Let

$$\begin{cases} g_1 = (1, 0, ..., \frac{\partial \phi}{\partial x_1}), \\ g_2 = (0, 1, ..., \frac{\partial \phi}{\partial x_2}), \\ ......, \\ g_n = (0, 0, ...1, \frac{\partial \phi}{\partial x_n}). \end{cases} \tag{1.13}$$

Define the Riemannian metric on $S(t)$:

$$(ds)^2 = \sum_{i,j=1}^{n} g_{ij}dx_idx_j,$$

where $g_{ij} = g_i \cdot g_j$. Clearly,

$$\begin{aligned} (ds)^2 &= \sum_{i=1}^{n}(1+\phi_{x_i}^2)(dx_i)^2 + \sum_{i\ne j, i,j=1}^{n}\phi_{x_i}\phi_{x_j}dx_idx_j \\ &= \sum_{i=1}^{n}(dx_i)^2 + \left(\sum_{i=1}^{n}\phi_{x_i}dx_i\right)^2. \end{aligned}$$

Recalling $\sum_{i=1}^{n} \phi_{x_i} dx_i = d\phi = dx_{n+1}$ for fixed $t > 0$, we get

$$(ds)^2 = \sum_{i=1}^{n+1} (dx_i)^2 = \sum_{i=1}^{n} (dx_i)^2 + (d\phi)^2.$$

If the derivatives $\frac{\partial \phi}{\partial x_i}$ are bounded for $i = 1, 2, ..., n$, then we can get a positive constant C, such that $\sum_{i=1}^{n} (\phi_{x_i} dx_i)^2 \leq C \sum_{i=1}^{n} (dx_i)^2$. Denoting $(d\rho)^2 = \sum_{i=1}^{n} (dx_i)^2$, which is the Euclidean metric on $\mathbb{R}^n$, we get

$$(d\rho)^2 \leq (ds)^2 \leq (1 + C)(d\rho)^2. \tag{1.14}$$

As a consequence of (1.14), we see that the completeness of $\mathbb{R}^n$ yields the completeness of $S(t)$ and therefore, $S(t)$ is a complete Riemannian manifold. On the other hand, if we can obtain

$$\nabla \left(\phi(x, t)\right) |_{\partial H_u(t)} = 0 \tag{1.15}$$

for every fixed $t > 0$, where $H_u(t)$ is the positivity set of $u(x, t)$:

$$H_u(t) = \{x \in \mathbb{R}^n : \ u(x, t) > 0\} \qquad\qquad t > 0,$$

then (1.15) encourage us to prove that: the manifold $S(t)$ is tangent to $\mathbb{R}^n$ on $\partial H_u(t)$.

It is well-known that the function

$$v(x, t) = \left(\frac{1}{2\sqrt{\pi t}}\right)^{-n} \int_{\mathbb{R}^n} u_0(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi$$

is the solution of the Cauchy problem of the linear heat equation (1.8) and $v(x, t) > 0$ in Q everywhere if only the initial value u_0 satisfies (1.2). This fact shows that the speed of propagation of $v(x, t)$ is infinite, that is to say,

$$\sup_{x \in H_v(t)} |x| = \infty. \tag{1.16}$$

However, the degeneracy of the equation in (1.1) causes an important phenomenon to occur, i.e. finite speed of propagation of disturbance. We have observed this phenomenon on the $source - type$ solution $B(x, t; C)$ (see [11]), where

$$B(x, t, C) = t^{-\lambda} \left(C - \kappa \frac{|x|^2}{t^{2\mu}}\right)_+^{\frac{1}{m-1}} \tag{1.17}$$

is the equation in (1.1) with a initial mass $M\delta(x)$, and

$$\lambda = \frac{n}{n(m-1)+2}, \qquad \mu = \frac{\lambda}{n}, \qquad \kappa = \frac{\lambda(m-1)}{2mn}.$$

We see that the function $B(x,t;C)$ has compact support in space for every fixed time. More precisely, if $u(x,t)$ is the solution of (1.1), (1.2), then

$$\sup_{x\in H_u(t)} |x| = O\left(t^{\frac{1}{n(m-1)+2}}\right) \tag{1.18}$$

when t is large enough (see Proposition 17 in [13]). Comparing (1.16) and (1.18) and recalling the mass conservation $\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx = \int_{\mathbb{R}^n} v(x,t)dx$, we will prove that the solution continuously depends on the nonlinearity of the equation (1.1):

$$\|u(\cdot,t) - v(\cdot,t)\|^2_{L^2(|x|\le k)} \le C\left[(m-1) + \frac{1}{k}\right].$$

We read the main conclusions of the present work as follows:

Theorem 1 *Assume $u(x,t)$ be the solution to (1.1), (1.2). Then $u(x,t) \in C(Q)$ and*

(1) for every given $\tau > 0, K > 0$, there exists a positive ν such that

$$|u(x_1,t_1) - u(x_2,t_2)| \le \nu\left(|x_1 - x_2|^{\frac{1}{h}} + |t_1 - t_2|^{\frac{1}{2h}}\right)$$

where $|x_i| \le K, t_i \ge \tau,\ i = 1,2,$

$$h = \begin{cases} 1 & if \quad 1 < m < 2, \\ h \in (m-1, m) & if \quad m \ge 2; \end{cases} \tag{1.19}$$

(2) for every $\beta > h$, the function $\phi = (u(x,t))^\beta \in C^1(Q)$ and the surface $\phi = \phi(x,t)$ is a complete Riemannian-manifold which is tangent to $\mathbb{R}^n$ on $\partial H_u(t)$ for every fixed $t > 0$, h is defined by (1.19);

(3) if $\beta > 2h$, the function $\phi(x,t)$ satisfies the degenerate parabolic equation

$$\frac{\partial\phi}{\partial t} = m\left[\phi^{\frac{m-1}{\beta}}\Delta\phi + \frac{m-\beta}{\beta}\phi^{\frac{m-\beta-1}{\beta}}|\nabla\phi|^2\right]$$

in the classical sense in Q.

Theorem 2 *Assume $u(x,t)$ be the solution to (1.1), (1.2), $B_\delta = \{x \in \mathbb{R}^n : \ |x| < \delta\}$ for some $\delta > 0$. If supp $u_0 \subset B_\delta$, then for every given $t > 0$,*

$$\sup_{x\in H_u(t)} |x| \ge \chi(t), \tag{1.20}$$

where,

$$\chi(t) = \left[(m-1)\pi^{\frac{(1-m)n}{2}}\Gamma(1+\frac{n}{2})^{m-1}\cdot\left(\int_{\mathbb{R}^n} u_0 dx\right)^{m-1} t\right]^{\frac{1}{2+(m-1)n}}.$$

Moreover, for every given $T > 0$, there is a positive $C_* = C_*(T)$ such that

$$\int_{|x|\leq k} [v(x,t) - u(x,t)]^2 \, dx \leq C_* \left[(m-1) + \frac{1}{k}\right] \tag{1.21}$$

with respect to $t \in (0,T)$ uniformly, where $v(x,t)$ is the solution of (1.8).

Let us also note that the manifolds $S(t)$ and $\mathbb{R}^n$ are two surfaces in $\mathbb{R}^{n+1}$ and the Cauchy problem (1.1), (1.2) can be regarded as a mapping $\Phi(t) : \mathbb{R}^n \longrightarrow S(t)$. Thus, besides the theorems mentioned above, we will give an example to show the intrinsic properties about the manifold $S(t)$.

2 The proof of Theorem 1

Lemma 1 *If $u(x,t)$ is the weak solution to (1.1), (1.2) in Q. Then for every $h \in (m-1, m)$, there is a $C_1 = C_1(h, m, M)$ such that*

$$\left|\nabla u^h\right|^2 \leq \frac{1}{C_1 t} \qquad \text{in } Q \tag{2.1}$$

in the sense of distributions in Q.

Proof: We first prove (2.1) for the classical solutions $u_\eta(x,t)$. Set

$$u_\eta^m = V^q \qquad \text{for } q \in \left(1, \frac{m}{m-1}\right).$$

Then

$$V_t = mV^{q-\frac{q}{m}} \Delta V + m(q-1)V^{q-1-\frac{q}{m}} |\nabla V|^2.$$

Differentiating this equation with respect to x_j and multiplying though by $\frac{\partial V}{\partial x_j}$, letting $h_j = \frac{\partial V}{\partial x_j}$, we get

$$\begin{aligned} \frac{1}{2}(h_j^2)_t &= mV^{q-\frac{q}{m}} h_j \Delta h_j + m(q - \frac{q}{m})V^{q-1-\frac{q}{m}} h_j^2 \Delta V \\ &\quad + m(q-1)(q-1-\frac{q}{m})V^{q-2-\frac{q}{m}} h_j^2 |\nabla V|^2 + 2m(q-1)V^{q-1-\frac{q}{m}} h_j \nabla V \cdot \nabla h_j \end{aligned}$$

for $j = 1, 2, ..., n$. Setting

$$H^2 = \sum_{j=1}^{n} h_j^2 \ ,$$

we obtain

$$\begin{aligned}
& H_t^2 - mV^{q-\frac{q}{m}}\Delta H^2 \\
= & \ 2m(q-\frac{q}{m})V^{q-1-\frac{q}{m}}H^2\Delta V + 2m(q-1)(q-1-\frac{q}{m})V^{q-2-\frac{q}{m}}\sum_{j=1}^{n} h_j^2|\nabla V|^2 \\
& +2m(q-1)V^{q-1-\frac{q}{m}}\nabla V\cdot\sum_{j=1}^{n}\nabla h_j^2 - 2mV^{q-\frac{q}{m}}\sum_{j=1}^{n}|\nabla h_j|^2 \\
\leq & \ 2m(q-\frac{q}{m})V^{q-1-\frac{q}{m}}H^2\Delta V + 2m(q-1)(q-1-\frac{q}{m})V^{q-2-\frac{q}{m}}H^4 \\
& +2m(q-1)V^{q-1-\frac{q}{m}}\nabla V\cdot\nabla H^2.
\end{aligned}$$

It follows from $q \in \left(1, \frac{m}{m-1}\right)$ that $(q-1)(q-1-\frac{q}{m}) < 0$ and $q-2-\frac{q}{m} < 0$. Hence,

$$2m(q-1)(q-1-\frac{q}{m})V^{q-2-\frac{q}{m}} \quad \leq \quad -\widetilde{C}_1,$$

where $\widetilde{C}_1 = 2m\left|(q-1)(q-1-\frac{q}{m})\right|(M+\eta)^{m-\frac{2m}{q}-1}$. Setting

$$\begin{aligned}
L(H^2) \quad = \quad & mV^{q-\frac{q}{m}}\Delta H^2 - \widetilde{C}_1 H^4 + 2m(q-\frac{q}{m})V^{q-1-\frac{q}{m}}H^2\Delta V \\
& +2m(q-1)V^{q-1-\frac{q}{m}}\nabla V\cdot\nabla H^2,
\end{aligned}$$

we get

$$H_t^2 \leq L(H^2).$$

Taking $h_1^* = (\frac{1}{\widetilde{C}_1 t})^{\frac{1}{2}}, h_i^* = 0$ for $i = 2,3,...,n$, and setting $Q_*^2 = \sum_{i=1}^{n} h_i^* = \frac{1}{\widetilde{C}_1 t}$, we see that the function Q_*^2 is a solution to the equation

$$\frac{\partial}{\partial t}Q_*^2 = L(Q_*^2)$$

with the initial condition $Q_*^2(0) = +\infty$.

On the other hand, let $u_\eta^m = K$, then K is the solution to the Cauchy problem of the linear parabolic equation

$$K_t = g(x,t)\Delta K \qquad\qquad \text{in } Q$$

with the inial value $(u_0+\eta)^m$, where, $g(x,t) = m(u_\eta(x,t))^{m-1}$ and $u_\eta(x,t)$ is a known function, which is the classical solution to (1.9) and $\eta \leq u_\eta \leq M+\eta$. By Theorem 5.1 in [18], we get $K \in H^{2+\alpha,1+\frac{\alpha}{2}}(\mathbb{R}^n\times(0,T))$ for any $T>0$ (Even if u_0 does not have the required smoothness we may approximate it (by mollification) with smooth functions $u_{0\eta}$). Therefore, ΔK, $\frac{\partial K}{\partial t}$ and $\frac{\partial K}{\partial x_i}(i = 1,2,...,n)$ are bounded. To be exact, there is a positive μ_0, which may depend on η, such that

$$|\Delta K| + |\frac{\partial K}{\partial t}| + |\nabla K| \leq \mu_0.$$

Therefore,

$$\begin{aligned} |\nabla V| &= |\nabla u_\eta^{\frac{m}{q}}| \\ &= |\frac{1}{q} u_\eta^{\frac{m}{q}-m} \nabla K| \\ &\leq \frac{1}{q} \eta^{(\frac{1}{q}-1)m} \mu_0. \end{aligned}$$

Letting $C' = \frac{1}{q}\eta^{(\frac{1}{q}-1)m}\mu_0$, we obtain $|\nabla V| \leq C'$. Similarly, we can get positive C'', which may depend on η also, such that $|\Delta V| \leq C''$.

Now we can employing the comparison theorem and get

$$H^2 \leq \frac{1}{\widetilde{C}_1 t}. \tag{2.2}$$

Letting $\eta \longrightarrow 0$ in (2.2) gives $|\nabla u^{\frac{m}{q}}|^2 \leq \frac{1}{C_1 t}$ for $q \in \left(1, \frac{m}{m-1}\right)$ with

$$C_1 = 2m \left|(q-1)(q-1-\frac{q}{m})\right| M^{m-\frac{2m}{q}-1}.$$

Setting $h = \frac{m}{q}$ in (2.2) yields $h \in (m-1, m)$, and (2.1) follows. $\square$

To prove Theorem 1, we need to show an ordinary inequality firstly:

$$|a-b|^\beta \leq |a^\beta - b^\beta| \qquad \text{for } a, b \geq 0,\ \beta > 1. \tag{2.3}$$

In fact, (2.3) is right for $a = b$. If $a > b$, we can easily get the following inequalities:

$$\left(1 - \frac{b}{a}\right)^\beta \leq 1 - \frac{b}{a} \qquad \text{and} \qquad 1 - \left(\frac{b}{a}\right)^\beta \geq 1 - \frac{b}{a}$$

thanks to $0 \leq \frac{b}{a} < 1$. Thereby, $\left(1 - \frac{b}{a}\right)^\beta \leq 1 - \left(\frac{b}{a}\right)^\beta$. This inequality gives

$$|a-b|^\beta = a^\beta \left|1 - \frac{b}{a}\right|^\beta \leq a^\beta - b^\beta.$$

So (2.3) holds for $a > b \geq 0$. Certainly, (2.3) is also right when $0 \leq a < b$.

We are now in a position to establish our Theorem 1.

To prove (1) of Theorem 1 It follows from (2.1) that

$$\left|u^h(x_1,t) - u^h(x_2,t)\right| \leq (C_1 t)^{-\frac{1}{2}} |x_1 - x_2| \tag{2.4}$$

for every $(x_1,t), (x_2,t) \in Q, h \in (m-1, m)$. If $1 < m < 2$, we take $h = 1$, and therefore, (2.4) yields

$$|u(x_1,t) - u(x_2,t)| \leq (C_1 t)^{-\frac{1}{2}} |x_1 - x_2|.$$

If $m \geq 2$, we take $h > 1$ for every $h \in (m-1, m)$. In this case, we use (2.3) in (2.4) and obtain

$$|u(x_1,t) - u(x_2,t)| \leq (C_1 t)^{-\frac{1}{2h}} |x_1 - x_2|^{\frac{1}{h}}.$$

Therefore, for every given $m > 1$, there always exists a suitable positive number $h \geq 1$ such that

$$|u(x_1,t) - u(x_2,t)| \leq (C_1 \tau)^{-\frac{1}{2h}} |x_1 - x_2|^{\frac{1}{h}} \tag{2.5}$$

for every $(x_1,t),(x_2,t) \in \mathbb{R}^n \times [\tau, \infty)$ with given $\tau > 0$, where $h = 1$ for $1 < m < 2$; $h \in (m-1, m)$ for $m \geq 2$. Employing the well-known theorem on the Hölder continuity with respect to the time variable (see [1]), we obtain

$$|u(x,t_1) - u(x,t_2)| \leq \mu |t_1 - t_2|^{\frac{1}{2h}} \tag{2.6}$$

for $|x| \leq K,\ t_1, t_2 > \tau$ and $|t_1 - t_2|$ small sufficiently, where μ depends on $(C_1\tau)^{-\frac{1}{2h}}, K$ is any fixed positive constant. Combining (2.5) and (2.6) we get another positive constant ν such that

$$|u(x_1,t_1) - u(x_2,t_2)| \leq \nu \left[|x_1 - x_2|^{\frac{1}{h}} + |t_1 - t_2|^{\frac{1}{2h}} \right] \tag{2.7}$$

for all $|x_i| \leq K, t_i \geq \tau,\ i = 1, 2$, where

$$h = \begin{cases} 1 & \text{if} \quad 1 < m < 2, \\ h \in (m-1, m) & \text{if} \quad m \geq 2. \end{cases}$$

Certainly, this gives $u \in C(Q)$. $\square$

To prove (2) of Theorem 1 Denote $\beta = h + \varepsilon$ and set

$$\phi(x,t) = u^{\beta}$$

for every $\varepsilon > 0$. It follows from (2.1) that

$$|\nabla \phi(x,t)| \leq C_2 u^{\varepsilon}(x,t) t^{-\frac{1}{2}} \qquad \text{in } Q \tag{2.8}$$

for some $C_2 > 0$. To prove the function $\phi(x,t) \in C^1(Q)$, we first see $u \in C^{\infty}(Q_+)$; second, (2.8) implies $\nabla\phi(x_*, t_*) = 0$ for every $(x_*, t_*) \in Q_0$. Combining (2.7) and (2.8) gives

$$\begin{aligned} |\nabla\phi(x,t) - \nabla\phi(x_*,t_*)| &= |\nabla\phi(x,t)| \\ &\leq C_2 t^{-\frac{1}{2}} \nu^{\varepsilon} \left[|x - x_*|^{\frac{1}{h}} + |t - t_*|^{\frac{1}{2h}} \right]^{\varepsilon}. \end{aligned} \tag{2.9}$$

This inequality tells us $\nabla\phi(x,t) \in C(Q)$ and therefore, $\phi(x,t) \in C^1(Q)$. Furthermore, for every fixed $t > 0$, the continuity of $u(x,t)$ implies $H_u(t)$ is a open set. Thus $\phi(x_*,t) = \nabla\phi(x_*,t) = 0$ for $(x_*,t) \in \partial H_u(t)$. This fact claims that the surface $\phi = \phi(x,t)$ touches $\mathbb{R}^n$ at $\partial H_u(t)$. In other words, $\mathbb{R}^n$ is just the tangent plane of $\phi(x,t)$ at $\partial H_u(t)$.

Now we can define a n-dimensional surface $S(t)$ for every fixed $t > 0$ as before:

$$S(t) : \begin{cases} x_i = x_i, & i = 1, 2, 3, ..., n, \\ x_{n+1} = \phi(x,t). \end{cases}$$

Defining the Riemannian metric (1.13) on $S(t)$, we get (1.14), more precisely,

$$(d\rho)^2 \le (ds)^2 \quad \le \quad \left(1 + \max_{i=1,2,\dots,n} \left|\frac{\partial \phi}{\partial x_i}\right|^2\right)(d\rho)^2.$$

It follows from (2.8) that $\max_{i=1,2,\dots,n} \left|\frac{\partial \phi}{\partial x_i}\right|^2 \le \left(C_2 u^{\varepsilon} t^{-\frac{1}{2}}\right)^2$. Moreover, as a result of theorem 9 in Chap III of [13], we get $u(x,t) \le C_3 t^{-\frac{n}{n(m-1)+2}}$ for some $C_3 > 0$. Hence,

$$\max_{i=1,2,\dots,n} \left|\frac{\partial \phi}{\partial x_i}\right|^2 \le C_4 t^{\frac{-2n\varepsilon}{n(m-1)+2}-1} \tag{2.10}$$

for another positive C_4. Thus,

$$(d\rho)^2 \le (ds)^2 \quad \le \quad \left(1 + C_4 t^{\frac{-2n\varepsilon}{n(m-1)+2}-1}\right)(d\rho)^2. \tag{2.11}$$

As a consequence of (2.11), the completeness of $\mathbb{R}^n$ yields the completeness of $S(t)$ and therefore, $S(t)$ is a complete Riemannian manifold. $\square$

To prove (3) of Theorem 1 Recalling $u(x,t) = \lim_{\eta \longrightarrow 0^+} u_\eta$ and $u_\eta \ge \eta$, u_η are the classical solutions to the Cauchy problem (1.9), we can make

$$\phi_\eta = u_\eta^\beta$$

for $\beta = h + \varepsilon > 2h$, and then ϕ_η satisfies the degenerate parabolic equation

$$\frac{\partial \phi_\eta}{\partial t} = m\left[\phi_\eta^{\frac{m-1}{\beta}} \Delta\phi_\eta + \frac{m-\beta}{\beta}\phi_\eta^{\frac{m-\beta-1}{\beta}} |\nabla\phi_\eta|^2\right] \qquad \text{in } Q.$$

Recalling $\lim_{\eta \longrightarrow 0^+} \phi_\eta = \phi$ and $u \in C^\infty(Q_+)$, we see that $\phi(x,t) \in C^\infty(Q_+)$ and ϕ satisfies the equation

$$\frac{\partial \phi}{\partial t} = m\left[\phi^{\frac{m-1}{\beta}} \Delta\phi + \frac{m-\beta}{\beta}\phi^{\frac{m-\beta-1}{\beta}} |\nabla\phi|^2\right] \tag{2.12}$$

in Q_+ . We next prove that (2.12) is also right in Q. To do this, we first see that (2.7) yields

$$|u(x,t) - u(x,t_*)| = u(x,t) \le \nu\, |t - t_*|^{\frac{1}{2h}} \qquad \text{for } (x,t_*) \in Q_0.$$

Thus, $|\phi(x,t) - \phi(x,t_*)| = \phi(x,t) \le \nu^\beta |t-t_*|^{\frac{\beta}{2h}}$. Thus,

$$\left|\frac{\phi(x,t) - \phi(x,t_*)}{t - t_*}\right| \le \nu^\beta\, |t - t_*|^{\frac{\beta-2h}{2h}}. \tag{2.13}$$

(2.13) shows the continuity of the function $\frac{\partial \phi}{\partial t}$, specially, $\frac{\partial \phi}{\partial t}|_{(x,t_*)} = 0$ owing to $\beta > 2h$. On the other hand, similar to (2.5) and (2.8), we can get

$$|u(x_1, ..., x_j, ..., x_n, t) - u(x_1, ..., x_{j*}, ..., x_n, t)| \le (C_1 t)^{-\frac{1}{2h}} |x_j - x_{j*}|^{\frac{1}{h}}$$

for every $(x_1, ..., x_j, ..., x_n, t), (x_1, ..., x_{j*}, ..., x_n, t) \in Q$ and $\left|\frac{\partial\phi}{\partial x_i}\right| \leq C_2 u^\varepsilon t^{-\frac{1}{2}}$ for $i = 1, 2, ..., n$. Thereby, if $(x_1, ..., x_{j*}, ..., x_n, t) \in Q_0$, we have

$$\begin{aligned} \left|\frac{\partial\phi(x_1, ..., x_j, ..., x_n, t)}{\partial x_i} - \frac{\partial\phi(x_1, ..., x_{j_*}, ..., x_n, t)}{\partial x_i}\right| &= \left|\frac{\partial\phi(x_1, ..., x_j, ..., x_n t)}{\partial x_i}\right| \\ &\leq C_2 t^{-\frac{1}{2}} (C_1 t)^{-\frac{\varepsilon}{2h}} |x_j - x_{j*}|^{\frac{\varepsilon}{h}}. \end{aligned}$$

This yields

$$\frac{1}{|x_j - x_{j*}|}\left|\frac{\partial\phi(x_1, ..., x_j, ..., x_n, t)}{\partial x_i} - \frac{\partial\phi(x_1, ..., x_{j_*}, ..., x_n, t)}{\partial x_i}\right| \leq C_2 (C_1 t)^{-\frac{\varepsilon}{2h}} |x_j - x_{j*}|^{\frac{\varepsilon}{h}-1}$$

for $i, j = 1, 2, ..., n$. This gives the continuity of the function $\frac{\partial^2\phi}{\partial x_i x_j}$ thanks to $\varepsilon > h$. In particular,

$$\frac{\partial^2\phi}{\partial x_i x_j} = 0 \qquad \text{on } Q_0$$

for $i, j = 1, 2, ..., n$. It follows from $m > 1$ that $\phi^{\frac{m-1}{\beta}} \Delta\phi = 0$ on Q_0. Similarly, $\phi^{\frac{m-\beta-1}{\beta}} |\nabla\phi|^2 = 0$ on Q_0.

Finally, as previously mentioned above, we deduce that the function $\phi(x, t)$ satisfies (2.12) on Q_0. □

As an applications of our Theorem 1, here we give an example to show the large time behavior on the intrinsic properties of the manifold $S(t)$.

Example (the first fundamental form on $S(t)$) In fact, the first fundamental form on the manifold $S(t)$ is $(ds)^2 = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$. By (2.10),

$$\begin{aligned} \left|(ds)^2 - (d\rho)^2\right| = (d\phi)^2 &= \left(\sum_{i=1}^{n} \phi_{x_i} dx_i\right)^2 \\ &\leq C_4 t^{-\frac{2n\varepsilon}{n(m-1)+2}-1} (d\rho)^2. \end{aligned}$$

where $(d\rho)^2 = \sum_{i,j=1}^{n} dx_i^2$ is just the first fundamental form on $\mathbb{R}^n$. Thus,

$$\frac{(ds)^2}{(d\rho)^2} = 1 + O\left(t^{t^{-\frac{2n\varepsilon}{n(m-1)+2}-1}}\right) \tag{2.14}$$

when t is large sufficiently.

3 The proof of Theorem 2

To prove Theorem 2, we need to establish a more precise Poincaré inequality. It is well-known that if λ_1 is the minimum positive eigenvalue and ψ_1 is the corresponding eigenfunction of the Dirichlet problem

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $\lambda_1 \|\psi_1\|^2_{L^2(\Omega)} = \|\nabla\psi_1\|^2_{L^2(\Omega)}$, where Ω is a boundary domain in $\mathbb{R}^n$. Moreover, if $\psi \in H^1_0(\Omega)$, then Poincaré inequality claims that there exists a positive constant k such that $k\|\psi\|^2_{L^2(\Omega)} \leq \|\nabla\psi\|^2_{L^2(\Omega)}$. According to Qiu (see p.98 in [21]), λ_1 is the maximum of all such k. We know that there are many kinds of choice for such k. For example, Wu (see p.13 in [23]) proved that

$$k \leq \rho^{-2}$$

if $\Omega = \{(x_1, x_2, ..., x_N) \in \mathbb{R}^N : \ a_i < x_i < a_i + \rho\ \}$. In order to prove Theorem 2 we need to show that such choice is also right if Ω is a sphere of $\mathbb{R}^n$.

Denote

$$B = \{x \in \mathbb{R}^n : \ |x - x_0| < \rho\}$$

for $x_0 \in \mathbb{R}^n$ and $\rho > 0$. We have the following result.

Lemma 2 *If $\Omega \subset B$, $u \in H^1(\Omega)$ and $u(x) = 0$ for $x \in \partial\Omega$. Then*

$$\|u\|_{L^2(\Omega)} \leq \rho\|\nabla u\|_{L^2(\Omega)} \ . \tag{3.1}$$

Proof We first suppose $u \in C_0^\infty(\Omega)$. For every $x \in \Omega$, there is a $x_* \in \partial\Omega$, such that the three points x_0, x and x_* lie on a radius $\overline{x_0\ x_*}$. Denote the vector from x_* to x by r. We have

$$\begin{aligned} u(x) &= u(x) - u(x_*) \\ &= \int_{x_*}^{x} \frac{\partial u(x)}{\partial r} dr. \end{aligned}$$

Using the Hölder inequality gets

$$|u(x)|^2 \leq \rho \int_{x_*}^{x} \left|\frac{\partial u(x)}{\partial r}\right|^2 dr.$$

Thus,

$$\begin{aligned} \int_\Omega |u(x)|^2 dx &\leq \rho \int_\Omega \int_{x_*}^{x} \left|\frac{\partial u(x)}{\partial r}\right|^2 dr dx \\ &\leq \rho^2 \int_\Omega |\nabla u(x)|^2 dx. \end{aligned}$$

The general case is done by approximation. $\square$

To prove (1.20) Assume $u(x,t)$ be the solution of (1.1), (1.2). Integrating (1.5) from t_1 to t_2 yields $\ln u(x,t_2) - \ln u(x,t_1) \geq -\frac{1}{m-1}(\ln t_2 - \ln t_1)$ for $t_2 > t_1$. This means $u(x,t_2)\cdot t_2^{\frac{1}{m-1}} \geq u(x,t_1)\cdot t_1^{\frac{1}{m-1}}$ for $t_2 > t_1 \geq 0$. Therefore,

$$\begin{cases} if\ u(x_0,t_0)=0,\ then\ u(x_0,t)=0\ \ for\ every\ \ 0\leq t<t_0; \\ if\ u(x_0,t_0)>0,\ then\ u(x_0,t)>0\ \ for\ every\ \ t>t_0. \end{cases} \tag{3.2}$$

By (3.2), we see that

$$H_u(t)\supset B_\delta \qquad\qquad t>0.$$

Absolutely, the proof is finished if $\sup\limits_{x\in H_u(t)}|x|=\infty$, otherwise, we set

$$K' = \gamma + \sup_{x\in H_u(t)}|x|$$

for $\gamma > 0$. Thus,

$$u(x,t)=0 \qquad\qquad \text{for } x\in\mathbb{R}^n - B_{K'}. \tag{3.3}$$

It follows from (1.5) that

$$\int_{B_{K'}} u^m\Delta u^m dx \geq -\frac{1}{(m-1)t}\int_{B_{K'}} u^{1+m}dx,$$

so that

$$\int_{B_{K'}} |\nabla u^m|^2 dx \leq \frac{1}{(m-1)t}\int_{B_{K'}} u^{1+m}dx.$$

Using (3.1) in this inequality, we obtain

$$\int_{B_{K'}} u^{2m} dx \leq \frac{K'^2}{(m-1)t}\int_{B_{K'}} u^{1+m}dx.$$

Employing the Hölder inverse inequality, we have

$$\int_{B_{K'}} u^{2m}dx \geq \left(\int_{B_{K'}} u^{1+m}dx\right)^{\frac{2m}{1+m}} |B_{K'}|^{\frac{1-m}{1+m}},$$

where $|B_{K'}|$ is the volume of $B_{K'}$ and $|B_{K'}| = \pi^{\frac{n}{2}}\Gamma(1+\frac{n}{2})^{-1}K'^n$. So we get

$$\left(\int_{B_{K'}} u^{1+m}dx\right)^{\frac{m-1}{1+m}} |B_{K'}|^{\frac{1-m}{1+m}} \leq \frac{K'^2}{(m-1)t}. \tag{3.4}$$

Using the Hölder inverse inequality again, we have

$$\int_{B_{K'}} u^{1+m}dx \quad \geq \quad \left(\int_{B_{K'}} udx\right)^{1+m} |B_{K'}|^{-m}. \tag{3.5}$$

It follows from (1.4) and (3.3) that $\int_{\mathbb{R}^n} u(x,t)dx = \int_{B_{K'}} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx$. Combining (3.4) and (3.5) we get

$$\left(\int_{\mathbb{R}^n} u_0dx\right)^{m-1} |B_{K'}|^{1-m} \leq \frac{K'^2}{(m-1)t}. \tag{3.6}$$

Now we get

$$(m-1)\left(\int_{\mathbb{R}^n} u_0dx\right)^{m-1} t \leq K'^{2+(m-1)n} \cdot \pi^{\frac{(m-1)n}{2}} \cdot \left(\Gamma(1+\frac{n}{2})\right)^{1-m}.$$

Letting $\gamma \longrightarrow 0$ gives

$$\sup_{x\in H_u(t)} |x| \geq \chi(t) \qquad\qquad t>0. \qquad\qquad \square$$

To prove (1.21) Assume $u(x,t)$ and $v(x,t)$ be the solutions to (1.1) and (1.8) respectively. Employing the well-known result (see [13]), we have

$$\frac{1}{1+m}\int_{\mathbb{R}^n} u^{1+m}(x,T)dx + \int_{Q_T} |\nabla u^m|^2 dxdt \leq \frac{1}{1+m}\int_{\mathbb{R}^n} u_0^{1+m}dx \tag{3.7}$$

and

$$\frac{1}{2}\int_{\mathbb{R}^n} v^2(x,T)dx + \int_{Q_T} |\nabla v|^2 dxdt \leq \frac{1}{2}\int_{\mathbb{R}^n} u_0^2dx \tag{3.8}$$

for every given $T>0$ and $Q_T = \mathbb{R}^n \times (0,T)$. Let

$$\begin{aligned} G &= v - u_\eta^m, \\ \psi &= \int_T^t G d\tau \qquad\qquad 0<t<T, \end{aligned}$$

where u_η are the solutions of (1.9). Let $\{\zeta_k\}_{k>1}$ be a smooth cutoff sequence with the following properties: $\zeta_k(x) \in C_0^\infty(\mathbb{R}^n)$ and

$$\zeta_k(x) = \begin{cases} 1 & |x| \leq k, \\ 0 < \zeta_k(x) < 1 & k < |x| < 2k, \\ 0 & |x| \geq 2k. \end{cases}$$

Clearly, there is a positive constant γ such that

$$|\nabla \zeta_k| \leq \frac{\gamma}{k} \qquad \text{and} \qquad |\Delta \zeta_k| \leq \frac{\gamma}{k^2}. \tag{3.9}$$

Recalling $(v-u_\eta)_t = \Delta G$ in Q_T, we multiply the equation by $\psi\zeta_k$ and integrate by parts in Q_T, we obtain

$$\int_{Q_T} (\zeta_k \nabla G \cdot \nabla\psi + \psi \nabla G \cdot \nabla \zeta_k) dxdt = \int_{Q_T} (v-u_\eta) G \zeta_k dxdt. \tag{3.10}$$

Differentiating (3.10) with respect to T, we get

$$\begin{aligned} \int_{\mathbb{R}^n} (v-u_\eta) G\zeta_k dx &= -\int_{Q_T} \left(\zeta_k |\nabla G|^2 + G\nabla G \cdot \nabla \zeta_k\right) dxdt \\ &\leq -\frac{1}{2}\int_{Q_T} \nabla G^2 \cdot \nabla \zeta_k dxdt. \end{aligned} \tag{3.11}$$

Letting $\eta \longrightarrow 0$ in (3.11), we get

$$\int_{\mathbb{R}^n} (v-u)(v-u^m)\zeta_k dx \leq \frac{\varphi}{k} \tag{3.12}$$

for some positive $\varphi = \varphi(T)$ thanks to (3.7), (3.8) and (3.9). On the other hand,

$$\int_{\mathbb{R}^n} (v-u) G\zeta_k dx = \int_{\mathbb{R}^n} (v-u)^2 \zeta_k dx + \int_{\mathbb{R}^n} (v-u)(u-u^m)\zeta_k dxdt.$$

Now we conclude that

$$\begin{aligned} \int_{\mathbb{R}^n} (v-u)^2 \zeta_k dx &\leq \int_{\mathbb{R}^n} |v-u| \cdot |u-u^m| \zeta_k dx + \frac{\varphi}{k} \\ &\leq \frac{1}{2}\int_{\mathbb{R}^n} \left[(v-u)^2 + (u-u^m)^2\right] \zeta_k dx + \frac{\varphi}{k}. \end{aligned}$$

This implies

$$\begin{aligned} \int_{\mathbb{R}^n} (v-u)^2 \zeta_k dx &\leq \int_{\mathbb{R}^n} (u-u^m)^2 \zeta_k dx + \frac{2\varphi}{k} \\ &\leq (m-1)\int_{\mathbb{R}^n} \xi^{m-1} |u-u^m| \zeta_k dx + \frac{2\varphi}{k} \\ &\leq (m-1)M^{m-1}\int_{\mathbb{R}^n} |u-u^m| \zeta_k dx + \frac{2\varphi}{k}. \end{aligned}$$

Recalling the definition of ζ_k, we see $\int_{\mathbb{R}^n} (v-u)^2 \zeta_k dx \geq \int_{|x|\leq k} (v-u)^2 dx$. Moreover, $\int_{\mathbb{R}^n} |u - u^m|\zeta_k dx$ is bounded thanks to (1.3) and (1.4). Hence, we can get a positive constant $C_* = C_*(T)$ such that

$$\int_{|x|\leq k} \left[v(x,t) - u(x,t)\right]^2 dx \leq C_* \left[(m-1) + \frac{1}{k}\right]$$

with respect to $t \in (0,T)$ uniformly. $\square$

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